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Separable states and positive maps

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Abstract

Using the natural duality between linear functionals on tensor products of C^*-algebras with the trace class operators on a Hilbert space H and linear maps of the C^*-algebra into $B(H)$, we study the relationship between separability, entanglement and the Peres condition of states and positivity properties of the linear maps.

Keywords: C^*-algebras; States; Positive maps

Introduction

In an earlier paper [14] we studied the duality between linear functionals $\tilde{\phi}$ on a tensor product $A \widehat{\otimes} \mathcal{T}$ of an operator system A and the trace class operators $\mathcal{T}$ on a Hilbert space H, and bounded linear maps $\phi : A \to B(H)$ given by the formula $\tilde{\phi}(a \otimes b) = \mathrm{Tr}(\phi(a)b^t)$. The main emphasis was on positivity properties of $\tilde{\phi}$ on cones in $A \widehat{\otimes} \mathcal{T}$ obtained by classes of positive maps. In the present paper we shall see how this study yields a natural framework for the study of separable states of $A \widehat{\otimes} \mathcal{T}$, for example we recover results of Horodecki et al. [9] and Horodecki, Shor and Ruskai [11] on characterizations of separable states. In addition we shall obtain characterizations of states on $A \widehat{\otimes} \mathcal{T}$ satisfying the Peres condition, viz. $\rho \circ (\iota \otimes t)$ is positive, where t is the transpose map and ι the identity map. In particular we see that nondecomposable maps yield natural examples of entangled states which satisfy the Peres condition; for this see also [7,8]. In the last section we study the definite set of a positive map ϕ on a C^*-algebra A, i.e. the set of self-adjoint operators in A such that $\phi(a^2) = \phi(a)^2$, and show that if $\tilde{\phi}$ is a separable state, then the

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image of the definite set is a C^*-subalgebra of the center of the C^*-algebra generated by $\phi(A)$. As a corollary we obtain a decomposition result for separable states in the finite-dimensional case.

1. Cones and states

In this section we recall notation and concepts from [14] and show a general characterization of separable states close to that in [11]. For more details on the following see [14].

By an *operator system* we shall mean a norm-closed self-adjoint set A of bounded operators on a Hilbert space containing the identity. We denote by $A \odot B(H)$ the algebraic tensor product of A and $B(H)$ and by $A \otimes B(H)$ the closure of $A \odot B(H)$ in the operator norm. If $\mathcal{T}$ denotes the trace class operators on H, then $A \widehat{\otimes} \mathcal{T}$ denotes the projective tensor product of A and $\mathcal{T}$. We denote by $B(A, H)$, (respectively $B(A, H)^+$) the bounded (respectively positive) linear maps of A into $B(H)$. The *BW-topology* on $B(A, H)$ is the topology of bounded pointwise weak convergence, i.e. a net (ϕ_ν) converges to ϕ if it is uniformly bounded, and $\phi_\nu(a) \to \phi(a)$ weakly for all $a \in A$. We denote by t the transpose map on $B(H)$ with respect to some orthonormal basis for H. Then by abuse of notation we get that the transpose map on $B(K) \otimes B(H)$ is $t \otimes t$. We shall also denote by Tr the usual trace on $B(H)$ which takes the value 1 on minimal projections. We recall Lemma 2.1 in [14].

Lemma 1. *With the above notation there is an isometric isomorphism $\phi \to \tilde{\phi}$ between $B(A, H)$ and $(A \widehat{\otimes} \mathcal{T})^*$ given by*

$$\tilde{\phi}(a \otimes b) = \operatorname{Tr}\bigl(\phi(a)b^t\bigr), \quad a \in A,\ b \in \mathcal{T}.$$

Furthermore, $\phi \in B(A, H)^+$ if and only if $\tilde{\phi}$ is positive on the cone $A^+ \widehat{\otimes} \mathcal{T}^+$ generated by operators of the form $a \otimes b$ with a and b positive.

We recall Definition 2.3 in [14]. It says that a BW-closed subcone $K \neq 0$ of $B(B(H), H)^+$ is a *mapping cone* if it has a BW-dense subset of ultra-weakly continuous maps and is invariant in the sense that if $\alpha \in K$, and $a, b \in B(H)$ then the map $x \to a\alpha(bxb^*)a^*$ belongs to K. Three mapping cones will be of special interest in the following, namely $B(B(H), H)^+$, $CP(H)$—the set of completely positive maps in $B(B(H), H)$, and $S(H)$—the BW-closed cone generated by maps of the form

$$x \to \sum_{i=1}^{n} \omega_i(x) a_i,$$

where ω_i is a normal state on $B(H)$ and $a_i \in B(H)^+$. The latter maps are said to be of "Holevo form" in [11]. By Lemma 2.4 in [14] $S(H)$ is the minimal mapping cone and $B(B(H), H)^+$ the maximal one.

If K is a mapping cone and A an operator system as before, we denote by $P(A, K)$ the cone

$$P(A, K) = \bigl\{x \in A \widehat{\otimes} \mathcal{T}\colon \iota \otimes \alpha(x) \geqslant 0\ \forall \alpha \in K\bigr\}.$$

By Lemma 2.8 in [14] $P(A,K)$ is a proper norm-closed convex cone in $A\mathbin{\widehat{\otimes}}\mathcal{T}$ containing the cone $A^+\mathbin{\widehat{\otimes}}\mathcal{T}^+$. A map $\phi\in B(A,H)$ is said to be *K-positive* if

$$\tilde{\phi}\Bigl(\sum a_i\otimes b_i\Bigr)=\sum \mathrm{Tr}\bigl(\phi(a_i)b_i^t\bigr)\geqslant 0\quad\text{whenever }\sum a_i\otimes b_i\in P(A,K).$$

By Theorem 3.2 in [14] ϕ is completely positive if and only if $\tilde{\phi}$ is positive on the cone $(A\mathbin{\widehat{\otimes}}\mathcal{T})^+$, the closure of the positive operators in $A\odot\mathcal{T}$, if and only if ϕ is $CP(H)$-positive.

If $C\subseteq V$ and $D\subseteq W$ are closed convex cones in two real locally convex vector spaces in duality, we denote by C^* (respectively D^*) the set of $w\in W$ such that $\langle v,w\rangle\geqslant 0$ $\forall v\in C$, (respectively $v\in V$ such that $\langle v,w\rangle\geqslant 0$ $\forall w\in D$). Thus ϕ is K-positive if and only if $\tilde{\phi}\in P(A,K)^*$. By a straightforward application of the Hahn–Banach theorem for closed convex cones, see e.g. [1, Proposition 1.32], we have

$$P(A,K)=P(A,K)^{**}.$$

We say a positive linear functional ρ on $A\otimes B(H)$ is *separable* if it belongs to the norm closure of positive sums of states of the form $\sigma\otimes\omega$, where σ is a state of A and ω a normal state of $B(H)$. Otherwise ρ is called *entangled*. We denote the set of separable states by $S(A,H)$. It is a norm-closed cone in $(A\mathbin{\widehat{\otimes}}\mathcal{T})^*$. As for P above $S(A,H)=S(A,H)^{**}$. Our next result is closely related to Theorem 2 in [11].

Theorem 2. *Let A be an operator system and $\phi\in B(A,H)$. Then the following conditions are equivalent*:

(i) *$\tilde{\phi}$ is a separable positive linear functional.*
(ii) *ϕ is $S(H)$-positive.*
(iii) *ϕ is a BW-limit of maps of the form $x\to\sum_{i=1}^n\omega_i(x)b_i$ with ω_i a state of A, and $b_i\in B(H)^+$.*

Proof. (i) $\Leftrightarrow$ (ii). Let Sn denote the positive normal linear functionals on $B(H)$, and let $x=\sum x_i\otimes y_i\in A\odot B(H)$. Then

$$\begin{aligned}
&x\in P\bigl(A,S(H)\bigr)\\
&\quad\Leftrightarrow\quad(\iota\otimes b\omega)(x)\geqslant 0\quad\forall\omega\in Sn,\ b\geqslant 0\\
&\quad\Leftrightarrow\quad\sum x_i\omega(y_i)\otimes b=\sum x_i\otimes\omega(y_i)b\geqslant 0\quad\forall\omega\in Sn,\ b\geqslant 0\\
&\quad\Leftrightarrow\quad\sum x_i\omega(y_i)\geqslant 0\quad\forall\omega\in Sn\\
&\quad\Leftrightarrow\quad\rho\otimes\omega(x)=\sum\rho(x_i)\omega(y_i)=\rho\Bigl(\sum x_i\omega(y_i)\Bigr)\geqslant 0\quad\forall\omega\in Sn,\ \rho\in A^{*+}\\
&\quad\Leftrightarrow\quad\eta(x)\geqslant 0\quad\forall\eta\in S(A,H)\\
&\quad\Leftrightarrow\quad x\in S(A,H)^*.
\end{aligned}$$

Thus ϕ is $S(H)$-positive if and only if $\tilde{\phi} \in P(A, S(H))^* = S(A, H)^{**} = S(A, H)$, proving that (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.6 in [14], since a map $\alpha \in S(H)$ if and only if $t \circ \alpha \circ t \in S(H)$. The proof is complete. □

In [11] maps like $x \to \sum \omega_i(x) b_i$ are called "entanglement breaking."

It is possible to give a direct proof of a less general form of the equivalence (i) $\Leftrightarrow$ (iii) above. Suppose $\phi(a) = \sum \omega_i(a) b_i$ for $a \in A$, $b_i \in B(H)^+$, ω_i state of A. Then

$$\tilde{\phi}(a \otimes b) = \mathrm{Tr}\big(\phi(a) b^t\big) = \sum \mathrm{Tr}\big(\omega_i(a) b_i b^t\big) = \sum \omega_i(a)\,\mathrm{Tr}\big(b_i b^t\big) = \sum \omega_i(a) \rho_i(b),$$

where $\rho_i(b) = \mathrm{Tr}(b_i b^t)$ is a positive linear functional. Thus $\tilde{\phi}$ is separable.

Conversely, if $\tilde{\phi} = \sum \omega_i \otimes \rho_i$, let $\tilde{\rho}_i(b) = \rho_i(b^t) = \mathrm{Tr}(b_i b)$. Then we have

$$\mathrm{Tr}\big(\phi(a) b^t\big) = \tilde{\phi}(a \otimes b) = \sum \omega_i(a) \rho_i(b) = \sum \omega_i(a) \tilde{\rho}_i\big(b^t\big) = \sum \mathrm{Tr}\big(\omega_i(a) b_i b^t\big).$$

This holds for all $b \in \mathcal{T}$, hence $\phi(a) = \sum \omega_i(a) b_i$.

Corollary 3. *Let H be separable and $\phi \in B(A, H)^+$. Suppose $\phi(A)$ is contained in an abelian C^*-algebra. Then $\tilde{\phi}$ is separable.*

Proof. By hypothesis there is an abelian von Neumann algebra $B \subseteq B(H)$ such that $\phi : A \to B$. Let (B_n) be an increasing sequence of finite-dimensional von Neumann subalgebras of B such that $\bigcup_n B_n$ is weakly dense in B. Let $E_n : B \to B_n$ be normal conditional expectations such that $E_{n-1}|_{B_n} \circ E_n = E_{n-1}$. Then $\phi(x) = \mathrm{wlim}_n E_n \circ \phi(x)$ for all $x \in A$. Since B_n is finite-dimensional, $E_n \circ \phi(x) = \sum \omega_i^n(x) e_i^n$, where ω_i^n are positive linear functionals on A and e_i^n are minimal projections in B_n. Since ϕ is a BW-limit of the sequence $E_n \circ \phi$, $\tilde{\phi}$ is separable by Theorem 2. The proof is complete. □

A celebrated necessary condition for a state ρ on $A \widehat{\otimes} \mathcal{T}$ to be separable is the *Peres condition*, i.e. $\rho \circ (\iota \otimes t) \geqslant 0$. A map $\phi \in B(A, H)$ is said to be *copositive* if $t \circ \phi$ is completely positive.

Proposition 4. *Let $\phi \in B(A, H)$. Then $\tilde{\phi}$ satisfies the Peres condition if and only if ϕ is both completely positive and copositive.*

Proof. If $a \in A$ and $b \in \mathcal{T}$ we have, since the trace is invariant under transposition,

$$\tilde{\phi}\big(a \otimes b^t\big) = \mathrm{Tr}\big(\phi(a) b\big) = \mathrm{Tr}\big(t \circ \phi(a) b^t\big) = (t \mathbin{\tilde{\circ}} \phi)(a \otimes b).$$

Thus $\tilde{\phi}$ satisfies the Peres condition if and only if both $t \mathbin{\tilde{\circ}} \phi$ and $\tilde{\phi}$ are positive. Using Theorem 3.2 in [14] this holds if and only if $t \circ \phi$ and ϕ are completely positive, hence if and only if ϕ is both completely positive and copositive. □

2. States on $B(K) \otimes B(H)$

In this section we study the case when the operator system A equals $B(K)$ for a Hilbert space K. But first we consider the finite-dimensional case. Let $M_n = M_n(\mathbb{C})$ denote the complex

$n \times n$ matrices, and let $\phi : M_n \to M_m$, so $\phi \in B(A, \mathbb{C}^m)$, where $A = M_n$ and $H = \mathbb{C}^m$. Let (e_{ij}) be a complete set of matrix units in M_n. Then the *Choi matrix* for ϕ is

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) = \iota \otimes \phi(P) \in M_n \otimes M_m,$$

where $\frac{1}{n}P$ is the 1-dimensional projection $\frac{1}{n}\sum e_{ij} \otimes e_{ij}$,—the so-called maximally entangled state, see [3]. Denote by ϕ^t the map $t \circ \phi \circ t$, where t denotes the transpose map in either M_n or in M_m. Then ϕ is completely positive if and only if ϕ^t is completely positive. It was shown by Choi [3] that ϕ is completely positive if and only if C_ϕ is positive. We use the convention that the density matrix for a state ρ is the positive matrix h such that $\rho(x) = \mathrm{Tr}(hx)$.

Lemma 5. *C_{ϕ^t} is the density matrix for $\tilde{\phi}$.*

Proof. Let $a \in M_n, b \in M_m$. Since the transpose t on $M_n \otimes M_m$ is the tensor product of the transpose operators on M_n and M_m, we have

$$\begin{aligned}
\mathrm{Tr}\big(C_{\phi^t} a \otimes b\big) &= \sum \mathrm{Tr}\big(e_{ij} \otimes \phi^t(e_{ij})(a \otimes b)\big) \\
&= \sum \mathrm{Tr}\big(e_{ji} \otimes \phi\big(e_{ij}^t\big)\big(a^t \otimes b^t\big)\big) \\
&= \sum \mathrm{Tr}\big(e_{ji} a^t\big) \mathrm{Tr}\big(\phi(e_{ji}) b^t\big) \\
&= \sum a_{ji} \mathrm{Tr}\big(e_{ji} \phi^*\big(b^t\big)\big) \\
&= \mathrm{Tr}\big(a \phi^*\big(b^t\big)\big) \\
&= \tilde{\phi}(a \otimes b).
\end{aligned}$$

In the above computation ϕ^* is the adjoint of ϕ as an operator between M_n and M_m considered as the trace class operators on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. The proof is complete. □

Lemma 6. *Let $H = \mathbb{C}^m$ and $\phi \in B(M_n, H)$. Then ϕ is positive if and only if $C_{\phi^t} \in P(M_n, S(H))$, if and only if $C_\phi \in P(M_n, S(H))$. Hence $P(M_n, S(H)) = \{C_\phi\colon \phi \geqslant 0\}$.*

Proof. By Theorem 2, or rather the proof of the equivalence (i) ⇔ (ii),

$$\begin{aligned}
&C_{\phi^t} \in P\big(M_n, S(H)\big) = S(M_n, H)^* \\
&\quad \Leftrightarrow \quad \mathrm{Tr}\big(C_{\phi^t} a \otimes b\big) \geqslant 0 \quad \forall a \in M_n^+,\ b \in M_m^+ \\
&\quad \Leftrightarrow \quad \phi \geqslant 0
\end{aligned}$$

by Lemma 1, proving the first statement. Since $\phi \geqslant 0$ if and only if $\phi^t \geqslant 0$, the above is equivalent to C_ϕ being in $P(M_n, S(H))$.

Each element $x \in P(M_n, S(H))$ defines a linear functional ρ on $M_n \otimes M_m$ by $\rho(y) = \mathrm{Tr}(xy)$. By Lemma 1 there is $\phi \in B(M_n, \mathbb{C}^m)$ such that $\rho(a \otimes b) = \mathrm{Tr}(\phi(a) b^t)$, hence by Lemma 5 and the first part of the proof, $x = C_{\phi^t}$ with $\phi \geqslant 0$. Thus the last statement follows, completing the proof. □

We shall now apply the finite-dimensional results to study states on $B(K) \otimes B(H)$ and to prove an infinite-dimensional extension of the Horodecki theorem [9]. Recall that a state and a positive linear map on a von Neumann algebra are said to be normal if they are weakly continuous on bounded sets.

Theorem 7. *Let ρ be a normal state on $B(K) \otimes B(H)$ with K and H Hilbert spaces and with density operator h. Then ρ is separable if and only if $\iota \otimes \psi(h) \geqslant 0$ for all normal positive maps $\psi : B(H) \to B(K)$.*

Proof. Suppose ρ is separable and normal. Then $\rho \circ (\iota \otimes \phi)$ is a normal state for all unital normal positive maps $\phi : B(K) \to B(H)$. Let ψ be as in the statement of the theorem. Then the adjoint map ψ^* is a positive map of the trace class operators on K into those on H. Thus if $x \geqslant 0$ is of finite rank in $B(K \otimes K) = B(K) \otimes B(K)$, then

$$\operatorname{Tr}\bigl((\iota \otimes \psi)(h)x\bigr) = \operatorname{Tr}\bigl(h\bigl(\iota \otimes \psi^*\bigr)(x)\bigr) = \rho\bigl(\iota \otimes \psi^*(x)\bigr) \geqslant 0,$$

hence $\iota \otimes \psi(h) \geqslant 0$.

To show the converse we first assume K and H are finite-dimensional. Then by Lemma 6 $P(M_n, S(H)) = \{C_\phi\colon \phi \geqslant 0\}$. Thus by Theorem 2 and Lemma 5 ρ is separable if and only if for all positive $\phi : B(K) \to B(H)$

$$\operatorname{Tr}\bigl(\bigl(\iota \otimes \phi^*\bigr)(h)P\bigr) = \operatorname{Tr}\bigl(h(\iota \otimes \phi)(P)\bigr) = \operatorname{Tr}(hC_\phi) \geqslant 0,$$

where P is the rank one matrix such that $C_\phi = \iota \otimes \phi(P)$. Since $P \geqslant 0$, and by assumption $\iota \otimes \phi^*(h) \geqslant 0$, it follows that ρ is separable.

We next consider the general case when K and H may be infinite-dimensional. Assume $\iota \otimes \psi(h) \geqslant 0$ for all normal $\psi : B(H) \to B(K)$. Since the maps $\psi_f(x) = \psi(fxf)$ are positive for all finite-dimensional projections f, it is clear that $\iota \otimes \psi((e \otimes f)h(e \otimes f)) \geqslant 0$ for all normal positive maps $\psi : B(H) \to B(K)$ with e a finite-dimensional projection in $B(K)$. Let

$$\psi_{e\otimes f}(y) = e\psi(fyf)e, \quad y \in B(H).$$

Then $\iota \otimes \psi_{e\otimes f}(h) \geqslant 0$. Now every normal positive map $\phi : B(fH) \to B(eK)$ is of the form $\psi_{e\otimes f}$ with ψ as above, because we can define $\phi : B(H) \to B(K)$ by $\psi(x) = \phi(fxf)$. Thus by the part of the proof on the finite-dimensional case, the positive linear functional $\omega(x) = \rho((e \otimes f)x(e \otimes f))$ is separable on $B(eK) \otimes B(fH)$. Since this holds for all finite-dimensional projections e and f and ρ is normal, it follows that ρ is separable. The proof is complete. □

We expect that the above theorem can be generalized to von Neumann algebras other than $B(K)$. If A is a semi-finite von Neumann algebra then so is $A \otimes B(H)$, hence each normal state on $A \otimes B(H)$ has a density operator with respect to a trace, and the formulation of the theorem has a natural generalization. In the type III case a formulation in terms of modular theory ought to be possible.

We next restate the Peres condition in terms of the density matrix of the normal state ρ.

Theorem 8. *Let ρ be a normal state on $B(K) \otimes B(H)$ with density operator h, and let t denote the transpose map of either $B(K)$ or $B(H)$. Then the following conditions are equivalent*:

(i) *ρ satisfies the Peres condition.*
(ii) $\iota \otimes t(h) \geqslant 0$.
(iii) $t \otimes \iota(h) \geqslant 0$.
(iv) $h \in P(B(K), CP(H)) \cap P(B(K), \mathrm{copos}(H))$*, where* $\mathrm{copos}(H)$ *denotes the copositive maps of* $B(H)$ *into itself.*

Proof. Assume (i). Since the trace on $B(K) \otimes B(H)$ is invariant under $\iota \otimes t$, we have

$$\rho \circ (\iota \otimes t)(a \otimes b) = \mathrm{Tr}\big(h(\iota \otimes t)(a \otimes b)\big) = \mathrm{Tr}\big(\iota \otimes t(h)(a \otimes b)\big).$$

Since $\rho \circ \iota \otimes t \geqslant 0$ it follows that $\iota \otimes t(h) \geqslant 0$.

Conversely, if (ii) holds then by the above computation $\rho \circ (\iota \otimes t) \geqslant 0$, hence (i) holds. The equivalence of (ii) and (iii) follows since $t \otimes \iota(h) = t \otimes t(\iota \otimes t(h))$, and the fact that $t \otimes t$ is an order-automorphism.

We have

$$\begin{aligned} P\big(B(K), \mathrm{copos}(H)\big) &= \big\{x \in B(K) \otimes B(H)\colon\ \iota \otimes \phi(x) \geqslant 0 \text{ for all copositive } \phi\big\} \\ &= \big\{x \in B(K) \otimes B(H)\colon\ \iota \otimes t(x) \geqslant 0\big\}, \end{aligned}$$

because a copositive map is the composition of a completely positive map and the transpose map. Thus (ii) is equivalent to (iv), completing the proof. □

Let A be a C^*-algebra. Then a map $\phi \in B(A, H)$ is called *decomposable* if it is the sum of a completely positive map and a copositive map. Otherwise ϕ is called *nondecomposable*. Since a map $\phi \in B(A, \mathbb{C}^n)$ is completely positive if and only if $\iota \otimes \phi : M_n \otimes A \to M_n \otimes M_n$ is positive [6, Lemma 5.1.3], it follows from [13] that $\phi \in B(A, \mathbb{C}^n)$ is decomposable if and only if whenever h and $t \otimes \iota(h)$ belong to $(M_n \otimes A)^+$ then $\iota \otimes \phi(h) \geqslant 0$. Thus ϕ is nondecomposable if and only if there exists $h \in (M_n \otimes A)^+$ such that $t \otimes \iota(h) \geqslant 0$ while $\iota \otimes \phi(h)$ is not positive. Suppose that $A = B(H)$, ϕ normal, and h as above. Then there exists by normality of ϕ a finite-dimensional projection $f \in B(H)$ such that $\iota \otimes \phi((1 \otimes f)h(1 \otimes f)$ is not positive. We can thus assume h is of finite rank. Normalizing h we thus have by Theorem 8 that the state $\rho(x) = \mathrm{Tr}(hx)$ satisfies the Peres condition, while by Theorem 7 ρ is entangled. We have thus proved

Theorem 9. *Let $\phi : B(H) \to M_n$ be normal positive and nondecomposable. Then there exists a normal state ρ on $B(H) \otimes M_n$ with density operator h such that $t \otimes \iota(h) \geqslant 0$, while $\iota \otimes \phi(h)$ is not positive. Hence ρ is entangled but satisfies the Peres condition.*

An explicit example of the situation in the above theorem is given in [13] and [5]. Then $\dim H = n = 3$, and $\phi : M_3 \to M_3$ is the nondecomposable Choi map [4]. Other examples can be found in [8] and [7]. A large class of nondecomposable maps are the projections onto spin factors of dimension greater than 6, or more generally, positive projections onto nonreversible Jordan algebras, see [12]. See [15] for another class of nondecomposable maps. Another result close to the above theorem can be found in [2]. Previous examples of entangled states which satisfy the Peres condition have been exhibited by P. Horodecki [10].

3. Definite sets

If A and B are C^*-algebras, and $\phi : A \to B$ is a positive map of norm $\leqslant 1$ then the (self-adjoint) *definite set* D_ϕ of ϕ is the set of self-adjoint operators in A such that $\phi(a^2) = \phi(a)^2$. D_ϕ is a Jordan algebra such that if $a \in D_\phi$ and $b \in A$ then $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ and $\phi(aba) = \phi(a)\phi(b)\phi(a)$, see [12]. We show in the present section that if ϕ is of the form $\phi(x) = \sum \omega_i(x)b_i$ as in Theorem 2, then $\phi(D_\phi)$ is contained in the center of the C^*-algebra generated by $\phi(A)$. In particular, if ϕ is faithful, then D_ϕ is abelian. As a consequence we get a decomposition result for separable states.

Theorem 10. *Let A be a unital C^*-algebra and $\phi \in B(A, H)^+$ with $\phi(1) = 1$. Suppose ϕ is of the form $\phi(x) = \sum_{i=1}^n \omega_i(x)b_i$ with ω_i states of A and $b_i \in B(H)^+$. Let e be a projection in the definite set D_ϕ of ϕ, and put $f = 1 - e$. Then $\phi(e)$ and $\phi(f)$ are projections in $B(H)$ and satisfy*

$$\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f)$$

for all $x \in A$. Hence $\phi(D_\phi)$ is an abelian C^*-algebra contained in the center of the von Neumann algebra generated by $\phi(A)$. In particular, if ϕ is faithful then D_ϕ is an abelian C^*-algebra.

Proof. Since $e \in D_\phi$, $\phi(e)$ and $\phi(f)$ are mutually orthogonal projections. Thus

$$0 = \mathrm{Tr}\big(\phi(e)\phi(f)\big) = \mathrm{Tr}\Big(\sum \omega_i(e)b_i\omega_j(f)b_j\Big) = \sum \omega_i(e)\omega_j(f)\,\mathrm{Tr}(b_ib_j).$$

Since each summand is positive we have

$$\omega_i(e)\omega_j(f)\,\mathrm{Tr}(b_ib_j) = 0 \quad \forall i, j.$$

In particular

$$\omega_i(e)\omega_i(f)\,\mathrm{Tr}\big(b_i^2\big) = 0 \quad \forall i.$$

Since $b_i \neq 0$ either $\omega_i(e) = 0$ or $\omega_i(f) = 0$ for all i. In particular, e or f belongs to the left and right kernel of ω_i, hence $\omega_i(exf) = \omega_i(fxe) = 0$ for all x. Thus $\omega_i(x) = \omega_i(exe) + \omega_i(fxf)$ for all x, so that

$$\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f),$$

where the last equality follows since $e, f \in D_\phi$.

To show the last statement in the theorem we consider the ultra-weakly continuous extension ϕ^{**} of ϕ to the second dual A^{**} of A. If $a \in D_\phi$ the abelian von Neumann algebra generated by a in A^{**} is contained in $D_{\phi^{**}}$ and is generated by its projections. It thus suffices to show that for each projection $e \in D_\phi$, $\phi(e)$ belongs to the commutant of $\phi(A)$. But this is immediate from the above equation.

If ϕ is faithful then the restriction of ϕ to D_ϕ is an isomorphism, hence is abelian, since $\phi(D_\phi)$ is abelian. The proof is complete. □

Corollary 11. *Let $A \subseteq B \subseteq B(H)$ be unital C^*-algebras with H separable. Suppose $\phi : B \to A$ is a conditional expectation. Then $\tilde{\phi}$ is separable if and only if A is abelian.*

Proof. By Corollary 3 if A is abelian then $\tilde{\phi}$ is separable. Since ϕ is a conditional expectation, the self-adjoint part of A equals the definite set D_ϕ, hence by Theorem 10 A is abelian if $\tilde{\phi}$ is separable, completing the proof. □

Let $\tilde{\phi} = \sum \lambda_i \omega_i \otimes \rho_i$ be a faithful separable state on $M_n \otimes M_m$, which is a convex sum of states ω_i on M_n and ρ_i on M_m. By symmetry in M_n and M_m in Lemma 1, there exists a completely positive map $\psi : M_m \to M_n$ such that $\tilde{\phi}(a \otimes b) = \mathrm{Tr}(a^t \psi(b))$. Since in particular the restrictions of $\tilde{\phi}$ to M_n and M_m are faithful, it follows that ϕ and ψ are faithful. Thus by Theorem 10 D_ϕ and D_ψ are abelian C^*-algebras. Let $(e_j)_{j=1,\dots,p}$ be minimal projections in D_ϕ and $(f_k)_{k=1,\dots,q}$ be minimal projections in D_ψ. From the proof of Theorem 10 the values of $\omega_i(e_j)$ and $\rho_i(f_k)$ are 0 or 1. In particular, the supports of ω_i and ρ_i are contained in some e_j and f_k, respectively. Hence $e_j \otimes f_k$ are mutually orthogonal projections with sum 1 such that

$$\tilde{\phi}(x) = \sum_{j,k} \tilde{\phi}\big((e_j \otimes f_k)x(e_j \otimes f_k)\big),$$

for all $x \in M_n \otimes M_m$.

We say $\tilde{\phi}$ is *irreducible* if $D_\phi = D_\psi = \mathbb{R}$ when we have cut down by the support of $\tilde{\phi}$, and we say a family (η_i) of states are *orthogonal* if their supports are mutually orthogonal. Summing up we have shown

Corollary 12. *Every separable state on $M_n \otimes M_m$ is a convex sum of orthogonal irreducible separable states.*

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